Muon Track Reconstruction

Summer Internship Report

INO Project

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Abstract

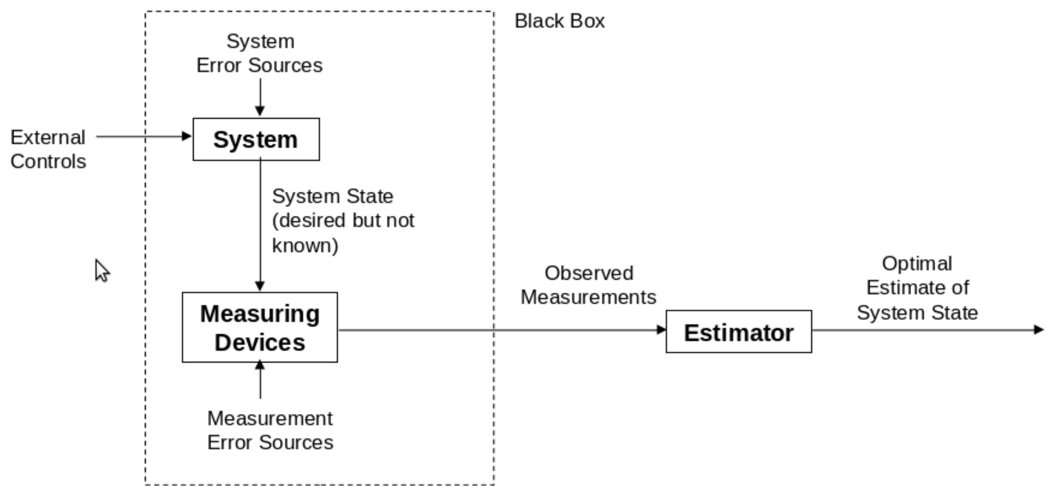
Tracking the path of a charged paticle is an essential task in high-energy physics. First, this report describes briefly the principles of KF (Kalman Filter), an efficient method for track fitting in HEP experiments. The next section includes discussion on momentum reconstruction of a typical monte carlo simulated muon based on Bethe-Bloch energy loss. Subsequently, in the report, the formulation for the theoretical prediction of the path of the muons and calculation related to extrapolation of the corresponding state vectors and **propagator jacobian** of the system for KF based track fitting have been investigated in order to find out any flaws which are giving rise to incorrect reconstructed momentum of the system.

Introduction:

The entire simulation framework in INO ICAL detecctor consists of four steps, namely **event generation**, **event simulation, event digitisation, event reconstruction**. Track reconstruction comes in the last one and consists of two steps: **track finding**, which selects hits belonging to a single track from the set of hits created by all the charged particles in an event, and **track fitting**, which fits the selected hits to a track model and determines its track parameters at the interaction point. For the fitting purpose, KF is used as usual as in other HEP experiments.

Principle of Kalman Filter:

The Kalman filter is a set of mathematical equations that provides an efficient com- putational (recursive) means to estimate the state of a process, in a way that mini- mizes the mean of the squared error. It deals with a system that is subject to a random disturbance (process noise) during its evolution following an equation of motion (system equation) the goal being extraction of the best estimate of this system's state at a given point from information collected at multiple observation points (measurement sites) .



The system equation is given by

$$x_k = f_{k-1}(x_{k-1}) + w_{k-1} = F_{k-1}x_{k-1} + w_{k-1} \tag{1}$$

where $f_{k-1}$is a non-linear function of state vector x_{k-1}, called **state propagator** and can be represented by a **propagator matrix** F_{k-1}.

$w_{k-1}$is termed as the **process noise.**

The measurement equation is given by

$$m_k = h_k(x_k) + \epsilon_k = H_k x_k + \epsilon_k \tag{2}$$

where h_k is called the **projector function** and can be represented as the projector matrix H_k .

ϵ_k is the random **measurement noise**. We assume that there is no bias in the either of the process noise and measurement noise, i.e.

$$< w_k >=< \epsilon_k >= 0 \tag{3}$$

The **noise covariances** are defined as $Q_k =< w_k w_k^T >$, $V_k =< \epsilon_k \epsilon_k^T >$ and the **estimation error covariance** as $C_k^i = x_k^i - x_k^0$ where x_k^i denotes predicted state vector of system at k^{th}site based on filtered data at i^{th} site. x_k^i is termed as **predicted** or **filtered** or **smoothed** state vector as $i < k$, $i = k$, $i > k$ respectively.

The state vector and estimation error covariance are predicted for k^{th}step based on $k-1^{th}$ step as

$$x_k^{k-1} = F_{k-1} x_{k-1}^{k-1} \tag{4}$$

and

$$C_k^{k-1} = F_{k-1} C_{k-1}^{k-1} F_{k-1}^T + Q_{k-1} \tag{5}$$

The χ^2 is constructed as

$$\chi^2 = [h_k(x_k^{k-1}) + H_k(x_k^k - x_k^{k-1}) - m_k]^T V_k^{-1} [h_k(x_k^{k-1}) + H_k(x_k^k - x_k^{k-1}) - m_k] \tag{6}$$

The x_k^k which minimizes the $\chi^2$is given by

$$x_k^k = x_k^{k-1} + [(C_k^{k-1})^{-1} + H_k^T V_k^{-1} H_k]^{-1} H_k^T V_k^{-1} (m_k - h_k(x_k^{k-1})) \tag{7}$$

The extrapolated estimation error covariance matrix comes out to be

$$C_k^k = [(C_k^{k-1})^{-1} + H_k^T V_k^{-1} H_k]^{-1} \tag{8}$$

The Kalman gain is defined as

$$K_k = [(C_k^{k-1})^{-1} + H_k^T V_k^{-1} H_k]^{-1} H_k^T V_k^{-1} = C_k^k H_k^T V_k^{-1} \tag{9}$$

So, finally the filtered state vector reads as

$$x_k^k = x_k^{k-1} + [(C_k^{k-1})^{-1} + K_k(m_k - h_k(x_k^{k-1})) \tag{10}$$

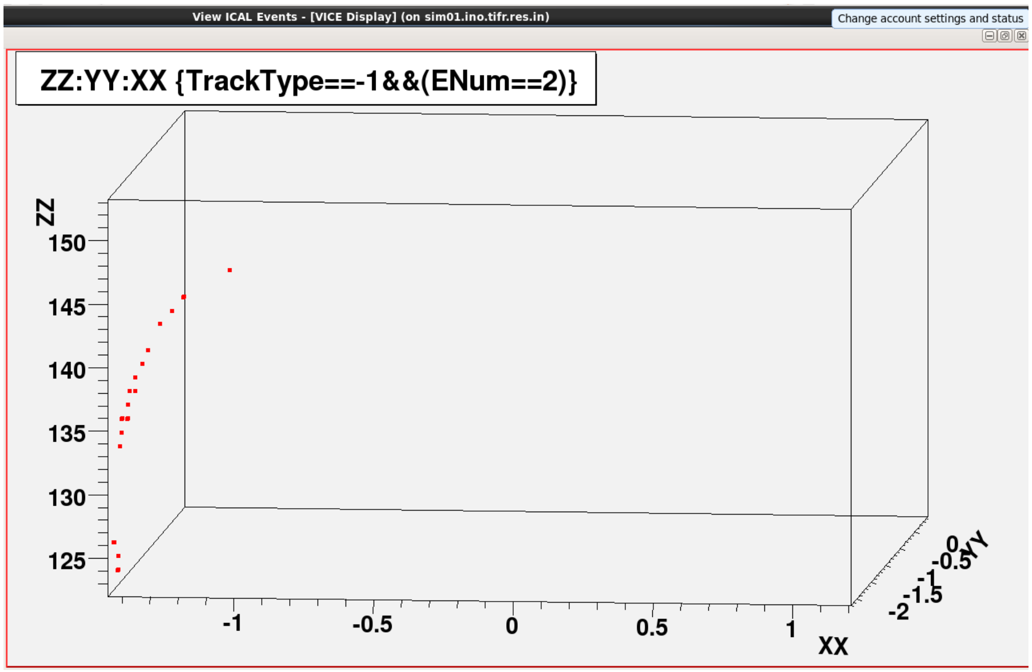
Muon Momentum Reconstruction:

In INO ICAL simulation, the monte carlo simulated muon tracks are investigated using the event display program VICE. Here is a typical example of the data from which initial momentum of the muon has been reconstructed based on the Bethe -Block Formula for energy loss. The ICAL code reconstructs the momentum from curvature formula, i.e.

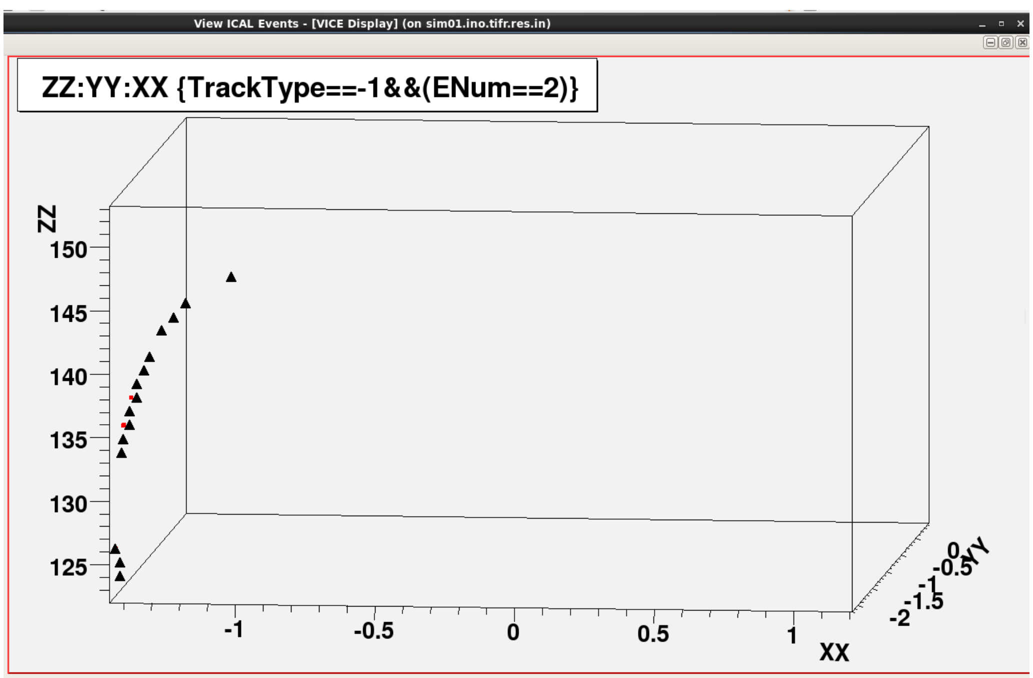
$$p = qBR$$

For muons whose path is completely within the entire ICAL detector (such as the one shown), the initial energy equals the energy lost by it to the calorimeter. But the energy deposited is easily calculable using Bethe-Bloch formula. Hence, initial momentum can be estimated.

Track finding (The red points are the hits from the muon only)



Track fitting (The black points show the filtered path of the muon)



Monte Carlo 2 GeV μ^- ($\cos\theta = 0.95$); ievt = 2

x (cm)	y (cm)	z (cm)	distance (cm)
-142.02	-214.62	1190.4	9.8
-142.02	-212.66	1200	9.99
-143.98	-210.7	1209.6	70.6
-143.98	-189.06	1276.8	10.37
-143.98	-185.14	1286.4	10.55
-142.02	-181.22	1296	9.8
-142.02	-179.26	1305.6	10.55
-140.06	-175.34	1315.2	9.8
-140.06	-173.38	1324.8	10.55
-138.1	-169.46	1334.4	9.99
-136.14	-167.5	1344	19.98
-132.22	-163.58	1363.2	10.55
-128.3	-161.62	1372.8	11.09
-124.38	-157.7	1382.4	25.48
-108.7	-151.82	1401.6	

'path length' within support structure=70.6 cm

Range=70.6 x 7.87 g/cm^2

E^{Fe} = Range x (dE/ds)
= 1111.22 MeV

Effective path length in RPC=158.51 cm x (56/96)

Range = 158.51 x (56/96) x 7.87 g/cm^2

E^{rpc}= Range x dE/ds
= 1455.36 MeV

So, this analysis shows that the reconstructed initial momentum of the muon (over 2.5 GeV) using Bethe-Bloch formula significantly differs from the true initial momentum (2 GeV). On the contrary, the curvature formula, which is used in present ICAL code, gives a better estimation of the momentum (about 1.88 GeV).

Track Extrapolation: Theoretical Prediction

Equation of motion of a charged particle-

The differential equation of motion of the negatively charged muon is given by Lorentz force equation

$$\frac{d\vec{p}}{dt} = \kappa q \, \vec{v} \times \vec{B} \tag{11}$$

where κ is a coefficient appearing due to choice of units. Now, magnetic force doesn't do work and hence $v = |\vec{v}|$ and $p = |\vec{p}|$ are constants. Hence the Lorentz equation reads

$$d\vec{p} = \kappa q \, \vec{v} \times B \, \vec{ds}/v \tag{12}$$

Introducing an unit vector $\vec{e} = \vec{v}/v = \vec{p}/p$, the Lorentz equation gives

$$d\vec{e} = \kappa . \frac{q}{p} . \vec{e} \times \vec{B} . ds = \kappa \frac{q}{p} . \begin{pmatrix} e_y B_z - e_z B_y \\ e_z B_x - e_x B_z \ . \\ e_x B_y - e_y B_x \end{pmatrix} ds \tag{13}$$

Now in ICAL detector the RPC are $||$ to $X - Y$ planes. So, it's convenient to express x and y coordinate of the muon as a function of z and describe the state of motion of the muon by the state vector $(\ x \quad y \quad t_x \quad t_y \quad \frac{q}{p} \ )$ where $t_x = \frac{dx}{dt}$ and $t_y = \frac{dy}{dt}$

It is clear that instead of z coordinates of the particle, if $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are mentioned, it fixes he shape of the path. $\frac{q}{p}$ then describes the motion instead of p

Now,

$$t_x = \frac{dx}{dz} = \frac{dx}{dt} / \frac{dz}{dt} = \frac{v_x}{v_z} = \frac{v_x}{v} / \frac{v_z}{v} = \frac{e_x}{e_z}$$

Similarly

$$t_y = \frac{e_y}{e_z}$$

So, calculating the differential track directions, we get

$$dt_x = \frac{de_x e_z - e_x de_z}{e_z^2} = \kappa . \frac{q}{p} . \frac{e_y e_z B_z - e_z^2 B_y - e_x^2 B_y + e_x e_y B_x}{e_z^2} . ds$$

$$= \kappa.\frac{q}{p}.(t_y.B_z - (1+t_x^2).B_y + t_x t_y.B_x).ds$$

(just by insertion of components of $d\vec{e}$ from modified Lorentz force equation)
Similarly

$$dt_y = \kappa.\frac{q}{p}.((1+t_y^2).B_x - t_x t_y.B_y + t_x.B).ds$$

Now, using the relation $ds = (1+t_x^2+t_y^2)^{1/2}.dz$, we get the differential equations of motion in terms of the state vectors

$$x^{'} = t_x \tag{14}$$

$$y^{'} = t_y \tag{15}$$

$$t_x^{'} = \kappa.\frac{q}{p}.(1+t_x^2+t_y^2)^{1/2}.(t_y.B_z - (1+t_x^2).B_y + t_x t_y.B_x) \tag{16}$$

$$t_y^{'} = \kappa.\frac{q}{p}.(1+t_x^2+t_y^2)^{1/2}.((1+t_y^2).B_x - t_x t_y.B_y + t_x.B).ds \tag{17}$$

$$(\frac{q}{p})^{'} = 0 \tag{18}$$

Extrapolation of State Parameters:

To proceed to extrapolation of the track we first define two vectors as:

$$\mathbf{a}(z) \equiv \kappa.\frac{q}{p}.\sqrt{1+t_x^2+t_y^2}.\left(\begin{array}{ccc} t_x t_y, & -(1+t_x^2), & t_y \end{array}\right) \tag{19}$$

$$\mathbf{b}(z) = \kappa.\frac{q}{p}.\sqrt{1+t_x^2+t_y^2}.\left(\begin{array}{ccc} (1+t_y^2), & -t_x t_y, & -t_x \end{array}\right) \tag{20}$$

The motivation of these definition comes from the differential equations for $t_x$and t_y. Next we notice the z dependence of the magnetic field. In ICAL mass the magnetic field at a particular (x, y) position doesn't vary as one moves from one layer of iron || to x-y plane to another. It is only a function of x and y coordinate. However, the x and y position of the muon themselves being function of z their is an implicit z dependence. That is:

$$\mathbf{B}(z) \equiv \mathbf{B}(x(z), y(z)) \equiv \left(\begin{array}{ccc} B_x(x(z), y(z)), & B_y(x(z), y(z)), & B_z(x(z), y(z)) \end{array}\right) \equiv \left(\begin{array}{cc} B_x(z), & B_y(z) \end{array}\right)$$

The next thing to consider is that $B_z = 0$ in ICAL experiment. So, finally

$$\mathbf{B}(z) \equiv (B_x(z), B_y(z))$$

Now, consider a generic function T with no explicit z dependence, like this:

$$T(z) \equiv T(t_x(z), t_y(z))$$

so that

$$T^{/}(z) = \frac{\partial T}{\partial t_x} t_x^{/}(z) + \frac{\partial T}{\partial t_y} t_y^{/}(z) = \sum_{i_1=x,y,z} B_{i_1}(z) \left(\frac{\partial T}{\partial t_x} a_{i_1}(z) + \frac{\partial T}{\partial t_y} b_{i_1}(z) \right) \quad (21)$$

Now, define:

$$T_{i_1}(z) \equiv \frac{\partial T}{\partial t_x} a_{i_1}(z) + \frac{\partial T}{\partial t_y} b_{i_1}(z) \quad (22)$$

so that we can write (19) in compact form

$$T^{/}(z) = \sum_{i_1=x,y,z} B_{i_1}(z) T_{i_1}(z)$$

It's easy to check that $T_{i_1}(z) \equiv T_{i_1}(t_x(z), t_y(z))$

Now, we recursively define

$$T_{i_1 \ldots i_k}(z) \equiv \frac{\partial T_{i_1 \ldots i_{k-1}}}{\partial t_x} a_{i_k}(z) + \frac{\partial T_{i_1 \ldots i_{k-1}}}{\partial t_y} b_{i_k}(z) \quad (23)$$

where obviously $T_{i_1 \ldots i_k}(z) \equiv T_{i_1 \ldots i_k}(t_x(z), t_y(z))$

With these definitions and skipping the calculation, we get to an analytic integral expression for $T(z_e)$ as follows:

$$T(z_e) = T(z_0) + \sum_{k=1}^{n} \sum_{i_1, \ldots, i_k = x,y,z} T_{i_1 \ldots i_k}(z_0). \left(\int_{z_0}^{z_e} \ldots \int_{z_0}^{z_{k-1}} B_{i_1}(z_1) \ldots B_{i_k}(z_k) dz_k \ldots dz_1 \right)$$

$$+ \sum_{i_1, \ldots, i_{k+1} = x,y,z} \int_{z_0}^{z_e} \ldots \int_{z_0}^{z_k} B_{i_1}(z_1) \ldots B_{i_{k+1}}(z_{k+1}) T_{i_1 \ldots i_{k+1}}(z_{k+1}) dz_{k+1} \ldots dz_1 \quad (24)$$

Now, if we replace T by $t_x$and t_y seperately, we get the analytic expressions:

$$t_x(z_e) = t_x(z_0) + \sum_{k=1}^{n} \sum_{i_1, \ldots, i_k = x,y,z} t_{x_{i_1 \ldots i_k}}(z_0). \left(\int_{z_0}^{z_e} \ldots \int_{z_0}^{z_{k-1}} B_{i_1}(z_1) \ldots B_{i_k}(z_k) dz_k \ldots dz_1 \right) \quad (25)$$

$$t_y(z_e) = t_y(z_0) + \sum_{k=1}^{n} \sum_{i_1, \ldots, i_k = x,y,z} t_{y_{i_1 \ldots i_k}}(z_0). \left(\int_{z_0}^{z_e} \ldots \int_{z_0}^{z_{k-1}} B_{i_1}(z_1) \ldots B_{i_k}(z_k) dz_k \ldots dz_1 \right) \quad (26)$$

and subsequently

$$x(z_e) = x(z_0) + \int_{z_0}^{z_e} t_x(z)dz \tag{27}$$

$$y(z_e) = y(z_0) + \int_{z_0}^{z_e} t_y(z)dz \tag{28}$$

Next, we define some quantities as magnetic field integrals as follows:

$$s_{i_1...i_k}(z_e) = \left(\int_{z_0}^{z_e} ... \int_{z_0}^{z_{k-1}} B_{i_1}(z_1)...B_{i_k}(z_k)dz_k...dz_1 \right) \tag{29}$$

$$S_{i_1...i_k}(z_e) = \int_{z_0}^{z_e} s_{i_1...i_k}(z)dz \tag{30}$$

and

$$h = \kappa.\frac{q}{p}.\sqrt{1 + t_x^2 + t_y^2} \tag{31}$$

$$A_{i_1...i_k} = t_{x_{i_1...i_k}}(z_0)/h^k \tag{32}$$

$$D_{i_1...i_k} = t_{y_{i_1...i_k}}(z_0)/h^k \tag{33}$$

With these definitions, the expressions look like:

$$x(z_e) = x(z_0) + (z_e - z_0)t_x(z_0) + \sum_{k=1}^{n} \sum_{i_1,...,i_k} h^k A_{i_1...i_k} S_{i_1...i_k} \tag{34}$$

$$y(z_e) = y(z_0) + (z_e - z_0)t_y(z_0) + \sum_{k=1}^{n} \sum_{i_1,...,i_k} h^k D_{i_1...i_k} S_{i_1...i_k} \tag{35}$$

$$t_x(z_e) = t_x(z_0) + \sum_{k=1}^{n} \sum_{i_1,...,i_k} h^k A_{i_1...i_k} s_{i_1...i_k} \tag{36}$$

$$t_y(z_e) = t_y(z_0) + \sum_{k=1}^{n} \sum_{i_1,...,i_k} h^k D_{i_1...i_k} s_{i_1...i_k} \tag{37}$$

where $s_{i_1...i_k}$, $S_{i_1...i_k}$, $A_{i_1...i_k}$,$D_{i_1...i_k}$ are needed to be calculated upto different orders required. We first calculate the 1st order field integrals which is okay for track finding purpose. First order field-integrals are calculated to be

$$s_{i_1} = (z_e - z_0)B_{i_1} \tag{38}$$

$$S_{i_1} = \frac{(z_e - z_0)^2}{2} B_{i_1} \tag{39}$$

and the first order analytic expressions of the state parameters are calculated to be

$$t_x(z_e) = t_x(z_0) + h(z_e - z_0)[t_x t_y B_x - (1 + t_x^2)B_y] \tag{40}$$

$$t_y(z_e) = t_y(z_0) + h(z_e - z_0)[(1 + t_y^2)B_x - t_x t_y B_y] \tag{41}$$

$$x(z_e) = x(z_0) + (z_e - z_0)t_x(z_0) + \frac{h}{2}(z_e - z_0)^2[t_x t_y B_x - (1 + t_x^2)B_y] \tag{42}$$

$$y(z_e) = y(z_0) + (z_e - z_0)t_y(z_0) + \frac{h}{2}(z_e - z_0)^2[(1 + t_y^2)B_x - t_x t_y B_y] \tag{43}$$

Calculation of Propagator Matrix (using analytic expression):

Once the equations for state vector are known, the next task is to calculate the propagator matrix F which is required to extrapolate the covariance matrix from one site to the next one. The propagator matrix is given by

$$F = \frac{d\vec{r}(z_e)}{d\vec{r}(z_0)} \equiv \begin{pmatrix} \frac{\partial x_e}{\partial x_0} & \frac{\partial x_e}{\partial y_0} & \frac{\partial x_e}{\partial t_{x_0}} & \frac{\partial x_e}{\partial t_{y_0}} & \frac{\partial x_e}{\partial (\frac{q}{p})_0} \\ \frac{\partial y_e}{\partial x_0} & \frac{\partial y_e}{\partial y_0} & \frac{\partial y_e}{\partial t_{x_0}} & \frac{\partial y_e}{\partial t_{y_0}} & \frac{\partial y_e}{\partial (\frac{q}{p})_0} \\ \frac{\partial t_{x_e}}{\partial x_0} & \frac{\partial t_{x_e}}{\partial y_0} & \frac{\partial t_{x_e}}{\partial t_{x_0}} & \frac{\partial t_{x_e}}{\partial t_{y_0}} & \frac{\partial t_{x_e}}{\partial (\frac{q}{p})_0} \\ \frac{\partial t_{y_e}}{\partial x_0} & \frac{\partial t_{y_e}}{\partial y_0} & \frac{\partial t_{y_e}}{\partial t_{x_0}} & \frac{\partial t_{y_e}}{\partial t_{y_0}} & \frac{\partial t_{y_e}}{\partial (\frac{q}{p})_0} \\ \frac{\partial (\frac{q}{p})_e}{\partial x_0} & \frac{\partial (\frac{q}{p})_e}{\partial y_0} & \frac{\partial (\frac{q}{p})_e}{\partial t_{x_0}} & \frac{\partial (\frac{q}{p})_e}{\partial t_{y_0}} & \frac{\partial (\frac{q}{p})_e}{\partial (\frac{q}{p})_0} \end{pmatrix}$$

which on a bit simplification reads

$$F = \begin{pmatrix} 1 & 0 & \frac{\partial x_e}{\partial t_{x_0}} & \frac{\partial x_e}{\partial t_{y_0}} & \frac{\partial x_e}{\partial (\frac{q}{p})_0} \\ 0 & 1 & \frac{\partial y_e}{\partial t_{x_0}} & \frac{\partial y_e}{\partial t_{y_0}} & \frac{\partial y_e}{\partial (\frac{q}{p})_0} \\ 0 & 0 & \frac{\partial t_{x_e}}{\partial t_{x_0}} & \frac{\partial t_{x_e}}{\partial t_{y_0}} & \frac{\partial t_{x_e}}{\partial (\frac{q}{p})_0} \\ 0 & 0 & \frac{\partial t_{y_e}}{\partial t_{x_0}} & \frac{\partial t_{y_e}}{\partial t_{y_0}} & \frac{\partial t_{y_e}}{\partial (\frac{q}{p})_0} \\ 0 & 0 & \frac{\partial (\frac{q}{p})_e}{\partial t_{x_0}} & \frac{\partial (\frac{q}{p})_e}{\partial t_{y_0}} & (1+\epsilon) \end{pmatrix}$$

For extrapolation of the state vector, an analytic expression of state vector at a site in terms of the state vector at the previous site is needed unlike Runge-kutta method which does the extrapolation numerically.

The calculated entries in the matrix are shown here:

$$\frac{\partial x(z_e)}{\partial t_x(z_0)} = (z_e - z_0) + \frac{1}{2}(z_e - z_0)^2 h \left[B_x t_y \left\{ 1 + \frac{t_x^2}{1 + t_x^2 + t_y^2} \right\} - B_y \left\{ 2t_x + \frac{t_x(1 + t_x^2)}{1 + t_x^2 + t_y^2} \right\} \right] \tag{44}$$

$$\frac{\partial x(z_e)}{\partial t_y(z_0)} = \frac{1}{2}(z_e - z_0)^2 h \left[B_x t_x \left\{ 1 + \frac{t_y^2}{1+t_x^2+t_y^2} \right\} - B_y \frac{t_y(1+t_x^2)}{1+t_x^2+t_y^2} \right] \tag{45}$$

$$\frac{\partial x(z_e)}{\partial(\frac{q}{p})_0} = \frac{1}{2}\kappa(1+t_x^2+t_y^2)^{\frac{1}{2}}(z_e - z_0)^2 \left\{ B_x t_x t_y - B_y(1+t_x^2) \right\} \tag{46}$$

$$\frac{\partial y(z_e)}{\partial t_x(z_0)} = \frac{1}{2}(z_e - z_0)^2 h \left[B_x \frac{t_x(1+t_y^2)}{1+t_x^2+t_y^2} - B_y t_y \left\{ 1 + \frac{t_x^2}{1+t_x^2+t_y^2} \right\} \right] \tag{47}$$

$$\frac{\partial x(z_e)}{\partial(\frac{q}{p})_0} = \frac{1}{2}\kappa(1+t_x^2+t_y^2)^{\frac{1}{2}}(z_e - z_0)^2 \left\{ B_x t_x t_y - B_y(1+t_x^2) \right\} \tag{48}$$

$$\frac{\partial y(z_e)}{\partial t_x(z_0)} = \frac{1}{2}(z_e - z_0)^2 h \left[B_x \frac{t_x(1+t_y^2)}{1+t_x^2+t_y^2} - B_y t_y \left\{ 1 + \frac{t_x^2}{1+t_x^2+t_y^2} \right\} \right] \tag{49}$$

$$\frac{\partial y(z_e)}{\partial t_y(z_0)} = (z_e - z_0) + \frac{1}{2}(z_e - z_0)^2 h \left[B_x \left\{ 2t_y + \frac{(1+t_y^2)t_y}{1+t_x^2+t_y^2} \right\} - B_y t_x \left\{ 1 + \frac{t_y^2}{1+t_x^2+t_y^2} \right\} \right] \tag{50}$$

$$\frac{\partial y(z_e)}{\partial(\frac{q}{p})_0} = \frac{1}{2}\kappa(1+t_x^2+t_y^2)^{\frac{1}{2}}(z_e - z_0)^2 \left\{ B_x(1+t_y^2) - B_y t_x t_y \right\} \tag{51}$$

$$\frac{\partial t_x(z_e)}{\partial t_x(z_0)} = 1 + (z_e - z_0) h \left[B_x t_y \left\{ 1 + \frac{t_x^2}{1+t_x^2+t_y^2} \right\} - B_y t_x \left\{ 2 + \frac{(1+t_x^2)}{1+t_x^2+t_y^2} \right\} \right] \tag{52}$$

$$\frac{\partial t_x(z_e)}{\partial t_y(z_0)} = (z_e - z_0) h \left[B_x t_x \left\{ 1 + \frac{t_y^2}{1+t_x^2+t_y^2} \right\} - B_y \frac{t_y(1+t_x^2)}{1+t_x^2+t_y^2} \right] \tag{53}$$

$$\frac{\partial t_x(z_e)}{\partial(\frac{q}{p})_0} = \kappa(1+t_x^2+t_y^2)^{\frac{1}{2}}(z_e - z_0) \left[B_x t_x t_y - B_y(1+t_x^2) \right] \tag{54}$$

$$\frac{\partial t_y(z_e)}{\partial t_x(z_0)} = (z_e - z_0) h \left[B_x \frac{t_x(1+t_y^2)}{1+t_x^2+t_y^2} - B_y t_y \left\{ 1 + \frac{t_x^2}{1+t_x^2+t_y^2} \right\} \right] \tag{55}$$

$$\frac{\partial t_y(z_e)}{\partial t_y(z_0)} = 1 + (z_e - z_0) h \left[B_x t_y \left\{ 2 + \frac{(1+t_y^2)}{1+t_x^2+t_y^2} \right\} - B_y t_x \left\{ 1 + \frac{t_y^2}{1+t_x^2+t_y^2} \right\} \right] \tag{56}$$

$$\frac{\partial t_y(z_e)}{\partial(\frac{q}{p})_0} = \kappa(1+t_x^2+t_y^2)^{\frac{1}{2}}(z_e - z_0) \left[B_x(1+t_y^2) - B_y t_x t_y) \right] \tag{57}$$

The equations mentioned above were also calculated previously and are hereby reinvestigated to be okay. However, the first two of the following derivatives were previously assumed to vanish which seems to be one source of error.

$$\frac{\partial\left(\frac{q}{p}\right)_{z_e}}{\partial t_x(z_0)} = -\frac{1}{\beta p_0 c}\frac{q}{p_0}\frac{\partial E}{\partial s}\frac{t_x}{(1+t_x^2+t_y^2)^{\frac{1}{2}}}\rho\, dz = -\frac{1}{\beta p_0 c}\frac{q}{p_0}\left(\frac{\partial E}{\partial s}ds\right)\frac{t_x}{1+t_x^2+t_y^2} \tag{58}$$

$$\frac{\partial\left(\frac{q}{p}\right)_{z_e}}{\partial t_y(z_0)} = -\frac{1}{\beta p_0 c}\frac{q}{p_0}\frac{\partial E}{\partial s}\frac{t_y}{(1+t_x^2+t_y^2)^{\frac{1}{2}}}\rho\, dz = -\frac{1}{\beta p_0 c}\frac{q}{p_0}\left(\frac{\partial E}{\partial s}ds\right)\frac{t_y}{1+t_x^2+t_y^2} \tag{59}$$

$$\frac{\partial\left(\frac{q}{p}\right)_{z_e}}{\partial\left(\frac{q}{p}\right)_{z_0}} = 1 - \frac{2}{q\beta c}\left(\frac{q}{p}\right)_0\frac{\partial E}{\partial s}ds \tag{60}$$

These modification gives better result, but not satisfactory. So, a bit different approach was taken to calculate these terms which are shown below:

We start considering the dissipative force F along with the Lorentz force F_{mag}.

$$\frac{d\vec{p}}{dt} = \vec{F}_{mag} + \vec{F} = \kappa q(\vec{v}\times\vec{B}) + \vec{F}$$

$$\Longrightarrow \frac{dp_i}{dt} = \kappa q(\vec{v}\times\vec{B})_i + F_i$$

$$\Longrightarrow dp_i = \kappa q(\vec{v}\times\vec{B})_i\frac{ds}{v} + F_i\frac{ds}{v}$$

B_z is 0 in here. So, the concerned equations are:

$$p_x^{n+1} = p_x^n - \kappa q.v_z.B_y\frac{ds}{v} + F_x\frac{ds}{v} = p_x^n - \kappa q.cos(\theta).B_y ds + F_x\frac{ds}{v} \tag{61}$$

$$p_y^{n+1} = p_y^n - \kappa q.v_z.B_x\frac{ds}{v} + F_y\frac{ds}{v} = p_y^n + \kappa q.cos(\theta).B_x ds + F_y\frac{ds}{v} \tag{62}$$

$$p_z^{n+1} = p_z^n - \kappa q.(v_xB_y - v_yB_x).\frac{ds}{v} + F_z\frac{ds}{v} = p_z^n + \kappa q.(t_xB_y - t_yB_x).cos(\theta).ds + F_z\frac{ds}{v} \tag{63}$$

Writing $cos(\theta).ds = dz$, we get,

$$p_x^{n+1} = p_x^n - \kappa q.B_y dz + F_x.\frac{ds}{v} \tag{64}$$

$$p_y^{n+1} = p_y^n + \kappa q.B_x dz + F_y.\frac{ds}{v} \tag{65}$$

$$p_z^{n+1} = p_y^n + \kappa q.(t_x B_y - t_y B_x).dz + F_z.\frac{ds}{v} \tag{66}$$

Now, the dissipative force comes from the Bethe-Block Energy loss $\frac{dE}{ds}$ and should be given by $\frac{dE}{ds}.\hat{v}$ i.e. $F_i = \frac{dE}{ds}.\frac{p_i}{p}$

So,

$$F_i\frac{ds}{v} = \frac{dE}{ds}.\frac{p_i}{p}.\frac{ds}{\beta c} = \frac{\partial E}{\partial i}\frac{E}{pc^2}\frac{p_i}{p}ds = \frac{\partial E}{\partial i}\frac{1}{\beta}\frac{p_i}{p}\rho dz \tag{67}$$

& hence

$$p_x^{n+1} = p_x^n - \kappa q.B_y dz + \frac{\partial E}{\partial x}\frac{1}{\beta}\frac{p_x}{p}\rho dz \tag{68}$$

$$p_y^{n+1} = p_y^n + \kappa q.B_x dz + \frac{\partial E}{\partial y}\frac{1}{\beta}\frac{p_y}{p}\rho dz \tag{69}$$

$$p_z^{n+1} = p_y^n + \kappa q.(t_x B_y - t_y B_x).dz + \frac{\partial E}{\partial z}\frac{1}{\beta}\frac{p_z}{p}\rho dz \tag{70}$$

Now,

$$\frac{\partial(\frac{q}{p})^{(n+1)}}{\partial t_i^{(n)}} = -\frac{q}{\{p^{(n+1)}\}^2}.\frac{\partial p^{(n+1)}}{\partial t_i^{(n)}}$$

$$= -\frac{q}{\{p^{(n+1)}\}^2}\frac{\partial}{\partial t_i^{(n)}}\left[\sqrt{\{p_x^{(n+1)}\}^2 + \{p_y^{(n+1)}\}^2 + \{p_z^{(n+1)}\}^2}\right]$$

$$= -\frac{q}{\{p^{(n+1)}\}^3}\left[p_x^{(n+1)}\frac{\partial p_x^{(n+1)}}{\partial t_i^{(n)}} + p_y^{(n+1)}\frac{\partial p_y^{(n+1)}}{\partial t_i^{(n)}} + p_z^{(n+1)}\frac{\partial p_z^{(n+1)}}{\partial t_i^{(n)}}\right] \tag{71}$$

&

$$\frac{\partial(\frac{q}{p})^{(n+1)}}{\partial(\frac{q}{p})^{(n)}} = \left(\frac{p^{(n)}}{p^{(n+1)}}\right)^2\frac{\partial p^{(n+1)}}{\partial p^{(n)}} = \left(\frac{p^{(n)}}{p^{(n+1)}}\right).\frac{1}{p^{(n+1)}}.\left[p_x^{(n+1)}\frac{\partial p_x^{(n+1)}}{\partial p^{(n)}} + p_y^{(n+1)}\frac{\partial p_y^{(n+1)}}{\partial p^{(n)}} + p_z^{(n+1)}\frac{\partial p_z^{(n+1)}}{\partial p^{(n)}}\right] \tag{72}$$

Using expressions for $p_i^{(n+1)}$ in terms of $p_i^{(n)}$, we get:

$$\frac{\partial(\frac{q}{p})^{(n+1)}}{\partial t_i^{(n)}} = -\frac{q}{\{p^{(n+1)}\}^3}[\left\{p_x^{(n)} - \kappa q B_y dz + \frac{\partial E}{\partial x}\frac{1}{\beta}\frac{p_x}{p}\rho dz\right\}\frac{\partial}{\partial t_i^{(n)}}\left\{p_x^{(n)} - \kappa q B_y dz + \frac{\partial E}{\partial x}\frac{1}{\beta}\frac{p_x}{p}\rho dz\right\}$$

$$+\left\{p_y^{(n)} + \kappa q B_y dz + \frac{\partial E}{\partial y}\frac{1}{\beta}\frac{p_y}{p}\rho dz\right\}\frac{\partial}{\partial t_i^{(n)}}\left\{p_y^{(n)} + \kappa q B_y dz + \frac{\partial E}{\partial y}\frac{1}{\beta}\frac{p_y}{p}\rho dz\right\}$$

$$+\left\{p_z^{(n)}+\kappa q\left(t_x^{(n)}B_y-t_y^{(n)}B_x\right)dz+\frac{\partial E}{\partial z}\frac{1}{\beta}\frac{p_z}{p}\rho dz\right\}\frac{\partial}{\partial t_i^{(n)}}\left\{p_z^{(n)}+\kappa q\left(t_x^{(n)}B_y-t_y^{(n)}B_x\right)dz+\frac{\partial E}{\partial z}\frac{1}{\beta}\frac{p_z}{p}\rho dz\right\}] \tag{73}$$

&

$$\frac{\partial(\frac{q}{p})^{(n+1)}}{\partial(\frac{q}{p})^{(n)}}=\left(\frac{p^{(n)}}{p^{(n+1)}}\right)^2\frac{\partial p^{(n+1)}}{\partial p^{(n)}}=\left(\frac{p^{(n)}}{p^{(n+1)}}\right).\frac{1}{p^{(n+1)}}.\left[p_x^{(n+1)}\frac{\partial p_x^{(n+1)}}{\partial p^{(n)}}+p_y^{(n+1)}\frac{\partial p_y^{(n+1)}}{\partial p^{(n)}}+p_z^{(n+1)}\frac{\partial p_z^{(n+1)}}{\partial p^{(n)}}\right]$$

$$=\frac{\{p^{(n)}\}^2}{\{p^{(n)}\}^3}[\left\{p_x^{(n)}-\kappa qB_ydz+\frac{\partial E}{\partial x}\frac{1}{\beta}\frac{p_x}{p}\rho dz\right\}\frac{\partial}{\partial p^{(n)}}\left\{p_x^{(n)}-\kappa qB_ydz+\frac{\partial E}{\partial x}\frac{1}{\beta}\frac{p_x}{p}\rho dz\right\}$$

$$+\left\{p_y^{(n)}+\kappa qB_ydz+\frac{\partial E}{\partial y}\frac{1}{\beta}\frac{p_y}{p}\rho dz\right\}\frac{\partial}{\partial p^{(n)}}\left\{p_y^{(n)}+\kappa qB_ydz+\frac{\partial E}{\partial y}\frac{1}{\beta}\frac{p_y}{p}\rho dz\right\}$$

$$+\left\{p_z^{(n)}+\kappa q\left(t_x^{(n)}B_y-t_y^{(n)}B_x\right)dz+\frac{\partial E}{\partial z}\frac{1}{\beta}\frac{p_z}{p}\rho dz\right\}\frac{\partial}{\partial t_i^{(n)}}\left\{p_z^{(n)}+\kappa q\left(t_x^{(n)}B_y-t_y^{(n)}B_x\right)dz+\frac{\partial E}{\partial z}\frac{1}{\beta}\frac{p_z}{p}\rho dz\right\}] \tag{74}$$

Writing $p_i=p.\frac{t_i}{\sqrt{1+t_x^2+t_y^2}}$,we get

$$\frac{\partial(\frac{q}{p})^{(n+1)}}{\partial t_x^{(n)}}=-\frac{q}{\{p^{(n+1)}\}^3}\times$$

$$[\left\{p_x^{(n)}-\kappa qB_ydz+\frac{\partial E}{\partial x}\frac{1}{\beta}\frac{p_x}{p}\rho dz\right\}\left\{\frac{p^{(n)}}{\sqrt{1+\{t_x^{(n)}\}^2+\{t_y^{(n)}\}^2}}+\frac{\partial E}{\partial x}.\frac{\rho}{\beta}dz-p^{(n)}t_x^{(n)}\frac{t_x^{(n)}}{[1+\{t_x^{(n)}\}^2+\{t_y^{(n)}\}^2]^{\frac{3}{2}}}\right\}$$

$$+\left\{p_y^{(n)}+\kappa qB_ydz+\frac{\partial E}{\partial y}\frac{1}{\beta}\frac{p_y}{p}\rho dz\right\}\left\{-p^{(n)}t_y^{(n)}\frac{t_x^{(n)}}{[1+\{t_x^{(n)}\}^2+\{t_y^{(n)}\}^2]^{\frac{3}{2}}}\right\}$$

$$+\left\{p_z^{(n)}+\kappa q\left(t_x^{(n)}B_y-t_y^{(n)}B_x\right)dz+\frac{\partial E}{\partial z}\frac{1}{\beta}\frac{p_z}{p}\rho dz\right\}\left\{-p^{(n)}\frac{t_x^{(n)}}{[1+\{t_x^{(n)}\}^2+\{t_y^{(n)}\}^2]^{\frac{3}{2}}}+\kappa qB_ydz\right\}] \tag{75}$$

$$\frac{\partial(\frac{q}{p})^{(n+1)}}{\partial t_y^{(n)}}=-\frac{q}{\{p^{(n+1)}\}^3}[\left\{p_x^{(n)}-\kappa qB_ydz+\frac{\partial E}{\partial x}\frac{1}{\beta}\frac{p_x}{p}\rho dz\right\}\left\{-p^{(n)}t_x^{(n)}\frac{t_y^{(n)}}{[1+\{t_x^{(n)}\}^2+\{t_y^{(n)}\}^2]^{\frac{3}{2}}}\right\}$$

$$+\left\{p_y^{(n)}+\kappa qB_ydz+\frac{\partial E}{\partial y}\frac{1}{\beta}\frac{p_y}{p}\rho dz\right\}\left\{\frac{p^{(n)}}{\sqrt{1+\{t_x^{(n)}\}^2+\{t_y^{(n)}\}^2}}+\frac{\partial E}{\partial y}.\frac{\rho}{\beta}dz-p^{(n)}t_y^{(n)}\frac{t_y^{(n)}}{[1+\{t_x^{(n)}\}^2+\{t_y^{(n)}\}^2]^{\frac{3}{2}}}\right\}$$

$$+\left\{p_z^{(n)}+\kappa q\left(t_x^{(n)}B_y-t_y^{(n)}B_x\right)dz+\frac{\partial E}{\partial z}\frac{1}{\beta}\frac{p_z}{p}\rho dz\right\}\left\{-p^{(n)}\frac{t_y^{(n)}}{[1+\{t_x^{(n)}\}^2+\{t_y^{(n)}\}^2]^{\frac{3}{2}}}-\kappa qB_ydz\right\}] \quad (76)$$

$$\frac{\partial(\frac{q}{p})^{(n+1)}}{\partial(\frac{q}{p})^{(n)}}=\left\{\frac{p^{(n)}}{p^{(n+1)}}\right\}^2[\left\{p_x^{(n)}-\kappa qB_ydz+\frac{\partial E}{\partial x}\frac{1}{\beta}\frac{p_x}{p}\rho dz\right\}\left\{\frac{t_x^{(n)}}{\sqrt{1+\{t_x^{(n)}\}^2+\{t_y^{(n)}\}^2}}+t_x^{(n)}\frac{\partial}{\partial p^{(n)}}\left[\frac{1}{\beta}\frac{\partial E}{\partial x}\right]\rho dz\right\}$$

$$+\left\{p_y^{(n)}+\kappa qB_ydz+\frac{\partial E}{\partial y}\frac{1}{\beta}\frac{p_y}{p}\rho dz\right\}\left\{\frac{t_y^{(n)}}{\sqrt{1+\{t_x^{(n)}\}^2+\{t_y^{(n)}\}^2}}+t_y^{(n)}\frac{\partial}{\partial p^{(n)}}\left[\frac{1}{\beta}\frac{\partial E}{\partial y}\right]\rho dz\right\}$$

$$+\left\{p_z^{(n)}+\kappa q\left(t_x^{(n)}B_y-t_y^{(n)}B_x\right)dz+\frac{\partial E}{\partial z}\frac{1}{\beta}\frac{p_z}{p}\rho dz\right\}\left\{\frac{1}{\sqrt{1+\{t_x^{(n)}\}^2+\{t_y^{(n)}\}^2}}+\frac{\partial}{\partial p^{(n)}}\left[\frac{1}{\beta}\frac{\partial E}{\partial z}\right]\rho dz\right\}] \quad (77)$$

Apparently these equations are okay to give the correct last row entries for the propagator matrix, however that is yet to check.

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