**PH 802 Quantum Mechanics**
Tutorial Sheet : 3
*(One Dimensional Problems & Simple Harmonic Oscillator)*

1. Using the uncertainty principle, evaluate the ground state energy of a harmonic oscillator of pulsation $\omega$ using the classical expression for its energy

$$E = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$$

2. The deuteron, the nucleus of deuterium is a bound state of a proton and neutron. It is assumed that the attractive interaction between the proton and the neutron, as a function of their mutual distance $r$, is sufficiently well represented by *Yukawa potential*

$$V(r) = -(g^2/r)\exp(-r/a)$$

where $g$ is the coupling constant of strong interaction and $a$ the range of nuclear forces which is of the order of 1 Fermi. Consider the internal energy of deuteron as the sum of the kinetic and potential energies. Using Heisenberg uncertainty principle, show that the binding energy is given by

$$E_d = -\frac{\hbar^2}{mr_d^2}\frac{a - r_d}{a + r_d}$$

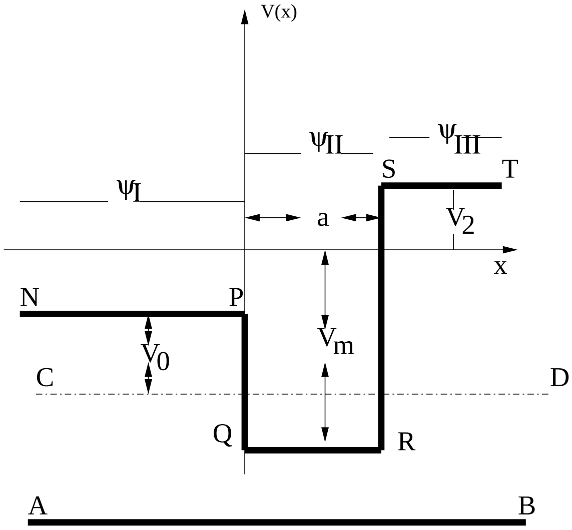
where $r_d$ is the radius of deuteron and $m$ is the mass of a nucleon (assume $m_n \approx m_p$. Using the experimentally known value of the binding energy to be $E_d = -2.2$ MeV, deduce the value of $a$

3. •An eigenfunction of $p_x$ must be an eigenfunction of the Hamiltonian $H$ but the eigenfunctions of $H$ need not be eigenfunctions of $p$. Here $p = -i\hbar(\partial/\partial x)$ and we restrict ourselves only to one dimension. Verify the above for (i) the free particle wavefunctions $\exp(ikx), \exp(-ikx), \sin x, \cos x$ (ii) the wave functions for particle in a box $\sin \alpha x$(iii) the ground state wavefunction of harmonic oscillator $\exp(-\beta x^2)$. Here $k, \alpha$ and $\beta$ are real positive constants. From the values of energy, interpret these constants.

4. Show that the 1 dimensional wavefunction $\exp(i\alpha x)$ where $\alpha$ is a real and positive constant represents a free particle moving towards right.(Why doesn't $\exp(\alpha x)$ do the same ?) Show this by comparing the full wave function $\exp(i\alpha x - iEt/\hbar)$ with the usual form $f(x - vt)$.

5. •Consider the wave function $\psi(x) = N\exp(-\alpha \mid x \mid)$. Calculate the normalization constant N. Calculate the probability of localization in an interval $(c - a/2, c + a/2)$ of width $a$, centered at the point $c$ on the abscissa. We want this probability to be 1/2. How should $c$ be chosen such that the interval $a$ becomes as small as possible ? What is the value of $a$ in this case ?

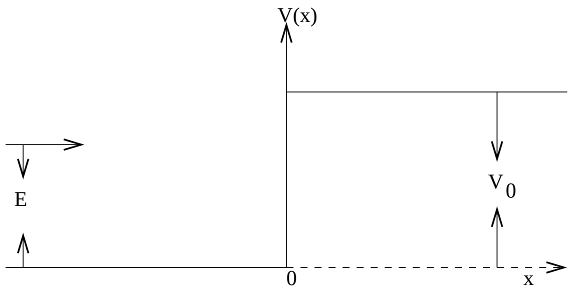
6. Show that no solution for quantum mechanical motion of a particle subjected to a one dimensional potential indicated by the thick curve is possible for the energy value of a particle indicated by the level AB. Deduce the energy quantization condition if the energy value is symbolically represented by CD.




7. Solve the Schrodinger equation for a particle in a one dimensional box taking the origin at its mid point and the ends at $\pm\frac{L}{2}$, where L is the length of the box.

8. • For a particle coming from left with an energy E on to a potential given by $V(x) = 0$ for $x < 0$; $V(x) = -\mid V_0 \mid$ for $x \geq 0$, obtain the reflection $R$ and transmission coefficients $(T)$. Show that $R + T = 1$.

9. A beam of particles with energy $E$ approaches, from the left, a potential step as shown. Find the value of $x(= x_0 > 0)$ for which the probability density is $1/e$ times the probability density at $x = 0$. Take the maximum allowed uncertainty $\Delta x$ for the particle to be localized in the classically forbidden region as $x_0$. Find the uncertainty this would cause in the energy of the particle.




10. • A particle of mass $m$ is bound in a square well of width $W$ and height $V_0$. The eigenfunction for the ground state is given by

$$\begin{aligned}\Psi(x) &= B\cos(\frac{4\pi x}{5W}) \qquad \text{for} \quad -\frac{W}{2} \leq x \leq \frac{W}{2} \\ &= A\exp(-\mid \alpha x \mid) \qquad \text{outside the well}\end{aligned}$$

(i) Taking $m$ and $W$ as given, find $\alpha, E$ and $V_0$. Compare the value of $E_1$ with that of infinite square well.

(ii) Find $A$ in terms of $B$ and given quantities.

(iii) Set up the integral from which $B$ can be calculated.

11. • It is desired to calculate the expectation value of the product of position $x$ and momentum $p_x$. Show that neither

$$\langle xp_x \rangle = \int_{-\infty}^{\infty} \psi^* x(-ih\frac{\partial}{\partial x})\psi dx$$

nor

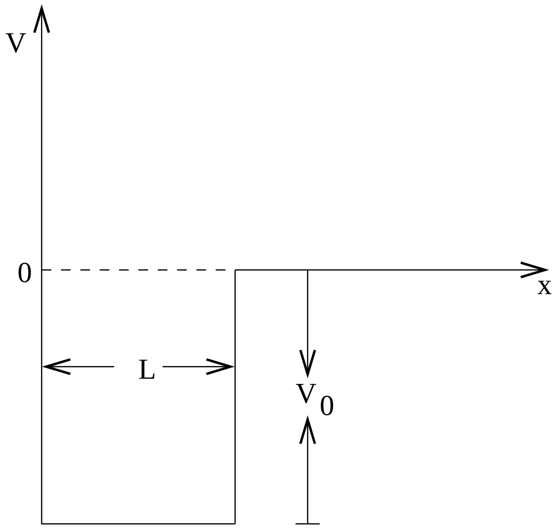
$$\langle p_x x \rangle = \int_{-\infty}^{\infty} \psi^* (-ih\frac{\partial}{\partial x}) x\psi dx$$

is acceptable, since both lead to imaginary values. Show that

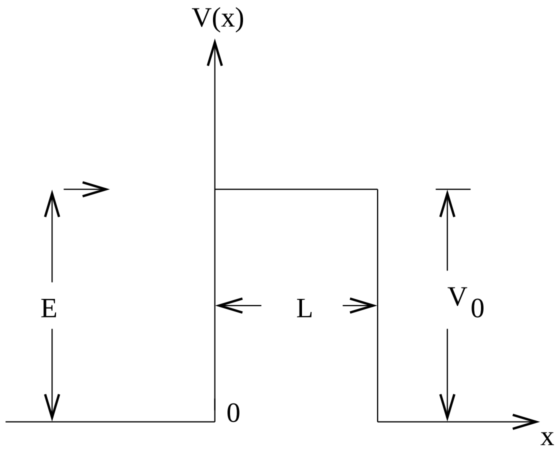
$$\langle x p_x \rangle = \frac{1}{2}\int_{-\infty}^{\infty} \psi^* [x(-ih\frac{\partial}{\partial x}) + (-ih\frac{\partial}{\partial x})x]\psi dx$$

leads to a real value.

12. •(a) For a particle coming from the right (from $x = \infty$) with energy $E > 0$, what is the reflection coefficient if the potential is as shown in the figure?
(b)For a given $L$, what would be $V_0$ so that there is only one bound state?




13. A beam of particles of energy $E$ and de Broglie wavelength $\lambda$ moving from left encounters a barrier as shown. If the barrier height $V_0 = E$, what is the transmission coefficient? Find the value of $L$ (in terms of $\lambda$) for which the reflection coefficient is 0.5.




14. • Given a potential well of depth $V_0$ and width $a$, draw the possible wave functions for the bound states for
    (i) $V_0 a^2 < \pi^2\hbar^2/8m$
    (ii) $\pi^2\hbar^2/8m < V_0 a^2 < 9\pi^2\hbar^2/8m$.

15. A particle coming in from the right with energy $V_0$, encounters a potential

$$\begin{aligned} V &= 0 \text{ for } x > a, \\ &= V_0 \text{ for } 0 < x \le a, \end{aligned}$$

$$= \quad -V_0 \text{ for } x \le 0.$$

(a) Write down the appropriate wave function in different regions (it is a linear function of x for $0 < x < a$).

(b) Obtain the transmission and reflection coefficients.

16. Consider a particle in a potential V(x) given by

$$\begin{aligned} V(x) &= 0 \text{ for } |x| < \frac{a}{2}, \\ &= V_0 \text{ for } |x| \ge \frac{a}{2}, \quad V_0 > 0. \end{aligned}$$

(a) Write down the appropriate even parity wavefunctions for $|x| < \frac{a}{2}$ and $x > \frac{a}{2}$.

(b) Obtain an implicit expression for the even parity bound state energy eigenvalues.

(c) Given that there is an even parity bound state with energy $E = \frac{V_0}{2}$, obtain the possible values of a in terms of $V_0$.

17. A particle of mass m is in a potential

$$\begin{aligned} V(x) &= 0 \text{ for } |x| < \frac{a}{2}, \\ &= \infty \text{ for } |x| \ge \frac{a}{2}. \end{aligned}$$

It is described by the wavefunction at time t=0 by

$$\psi(x,0) = A[cos(\frac{\pi x}{a}) + sin(\frac{2\pi x}{a})]$$

(a) Determine the normalization constant A.

(b) Obtain the average energy E.

(c) Write down the wavefunction at time t.

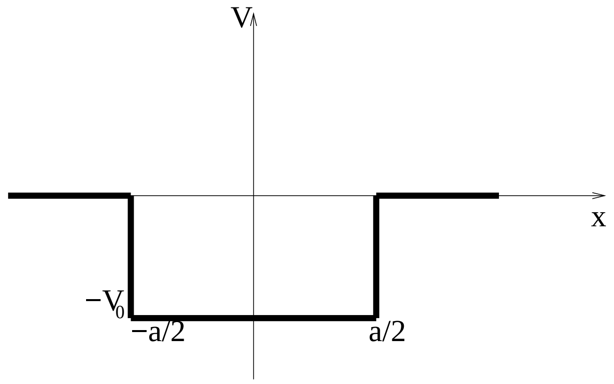
(d) Calculate the average value of x at time t.

18. • A particle of mass $m$ is under the influence of a potential $V(x) = \frac{1}{2}m\omega^2 x^2$ where $x$ is the displacement from the origin and $\omega$ is a constant. Show that $\psi(x) = Ax\exp(-\frac{m\omega}{2\hbar}x^2)$ is a solution of the one dimensional Schrodinger equation with the above potential. Find the value of energy for this state.

19. For a particle in a potential of the type shown in the figure

(a) What is the condition on $V_0$ and $a$ in terms of known constants that it willl have 7 bound states.

(b) What is the parity and the limits on the energy of the least bound state?



20. Show that the Hermite polynomials are generated in the Taylor series expansion

$$exp(2yt - t^2) = \sum_{n=0}^{\infty} \frac{H_n(y)}{n} t^n$$

21. Show that $H_n$ are also generated by

$$H_n(y) = (-1)^n (e^{y^2} \frac{\partial^n}{\partial y^n} e^{-y^2})$$

22. Use any of the preceding relations to show

$$H_n' = 2nH_{n-1}$$

23. Prove the recursion relation $H_{n+1} = 2yH_n - 2nH_{n-1}$

24. Use the generating function to generate $H_0, H_1 H_2$ and $H_4$

25. The general formula for the normalisation constant for the n$^{th}$ state eigenfunction is

$$A_n = (2^n n\sqrt{\pi})^{-1/2}$$

Show that this gives correct normalisation for $\phi_4$.

26. Show that the normalised n$^{th}$ eigenstate $\phi_n$ is generated from the normalized ground state $\phi_0$ through

$$\phi_n = \frac{1}{\sqrt{n}}(a^\dagger)^n \phi_0$$

27. Show that in the n$^{th}$ eigenstate of the harmonic oscillator, the avearge kinetic energy $< T >$ is equal to the average potential energy $< V >$ (the Virial Theorem)

28. Write down the Hamiltonian of a simple harmonic oscillator in the momentum space. Can you guess the form of the eigenstates in the momentum state.

29. If $\psi(x)$ is even and $\phi(x)$ is odd under $x \to -x$, show that

$$\int_{-\infty}^{\infty} \psi(x)\phi(x)dx = 0$$

Use this to show that $\psi_2(x)$ and $\psi_1(x)$ are orthogonal. Also verify that $\psi_2(x)$ and $\psi_0(x)$ are orthogonal.

30. Show that

(a)

$$< n'|X|n >= (\frac{\hbar}{2m\omega})^{1/2}[\delta_{n'}\delta_{n+1}(n+1)^{\frac{1}{2}} + \delta_{n'}\delta_{n-1}n^{\frac{1}{2}}]$$

(b)

$$< n'|P|n >= (\frac{m\omega\hbar}{2})^{1/2}\imath[\delta_{n'}\delta_{n+1}(n+1)^{\frac{1}{2}} - \delta_{n'}\delta_{n-1}n^{\frac{1}{2}}]$$

31. Compute the matrix elements of X and P in the $|n\rangle$ basis.

32. Find

    (a) $\langle X \rangle$

    (b) $\langle P \rangle$

    (c) $\langle X^2 \rangle$

    (d) $\langle P^2 \rangle$

    (e) $\langle n|X^4|n \rangle$

33. Problem Number 7.4.9 from Shankar's book.

34. Project the equation $a|0\rangle = 0$ on the p basis and obtain $\psi_0(p)$.

35. Project the relation $a|n\rangle = n^{\frac{1}{2}}|n-1\rangle$ on the X basis and derive the recursion relation $H_n'(y) = 2nH_{n-1}(y)$

36. Starting with $\psi_0(x)$ obtain $\psi_2(x)$.