PH 802 Quantum Mechanics

Tutorial Sheet 1
(to be attempted on Thursday August 12, 2004)

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1. Calculate the total emissive power of a black body at the following temperatures
   (a) 1K (b) 300K (c) 2000K (d) 4000K
   Find in each case the wavelength λ_{max} at which the spectral emittance has its maximum value.

2. Obtain Raleigh Jeans law as the long wavelength limit of the Planck radiation law.

3. Using Plank's distribution law to prove that $\lambda_{max}T = \frac{hc}{4.965\ k_B}$

4. Use Plank's distribution law to get the value of Stefan's constant.

5. The Photoelectric work function W of lithium is 2.3 eV.

   (a) Find the threshold frequency ν_t and the corresponding wavelength λ_t.

   (b) If ultraviolet light of wavelength $\lambda = 3000$ A° is incident on a lithium surface, calculate the maximum kinetic energy of the photoelectrons and the value of the stopping potential.

6. Consider the scattering of a photon by a free electron initially at rest. Find

   (a) The angle of scattering ϕ of electron in terms of incident waelength, scattered wavelength and the angle θ.

   (b) The kinetic energy of the scattered electron.

7. Show that a free photon cannot produce e^+e^- pair.

8. Calculate the threshold for the reaction (mass of proton = 938 MeV and mass of eta meson = 547 MeV)

   $\gamma + p \to p + \eta$

9. In Bohr model for hydrogen atom calculate the speed of electron moving in the first orbit, second orbit. Also calculate the angular momentum.

10. How does Bohr model gets modified if one does not assume that the nucleus is infinitely heavy.

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11. Consider the set of all entities of the form (a, b, c) where the entries are real numbers. Addition and scalar multiplication are defined as follows

    $(a, b, c) \; + \; (d, e, f) \; = \; (a + d, b + d, c + f)$
    $\alpha(a, b, c) = (\alpha a, \alpha b \alpha c).$

    Write down the null vector and inverse of (a,b,c). Do vectors of the form (a,b,1) form a vector space?

12. Prove the following properties of the vector space

    $|0\rangle$ is unique.

    $0|V\rangle = |0\rangle$

    $|-V\rangle = -|V\rangle$

    $|-V\rangle$ is a unique additive invers of $|V\rangle$

13. Consider three elements from the vector space of real 2×2 matrices:

    $$|1\rangle = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \quad |2\rangle = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad |3\rangle = \begin{bmatrix} -2 & -1 \\ 0 & -2 \end{bmatrix}$$

    Are they linearly independent?

14. Show that the following row vectors are linearly dependent : (1,1,0), (1,0,1) and (3,2,1). Show the opposite for (1,1,0), (1,0,1) and (0,1,1).

15. Justify the statement that *Any vector $|V\rangle$ in an n dimesional space can be written as a linear combination of n linearly independent vectors*

16. What is the dual of the ket

    $|V\rangle = (2 - 3\imath)|V_1\rangle + 5e^{\imath\pi/4}|V_2\rangle$

    where $|V_1\rangle$ and $|V_2\rangle$ are orthogonal pair of kets? Find the norm of $|V\rangle$

17. $|V_1\rangle$ and $|V_2\rangle$ are two normalized kets such that $\langle V_1|V_2\rangle \; = \; \imath/2$ Define

    $|X\rangle \; = \; 2|V_1\rangle + (2 - 3\imath)|V_2\rangle$
    $|W\rangle \; = \; (3 - 2\imath)|V_1\rangle + 2|V_2\rangle$

    Find the norms of $|X\rangle$ and $|W\rangle$ and determine $\langle X|W\rangle$ and $\langle W|X\rangle$.

18. Consider three kets $|e_1\rangle, |e_2\rangle$ and $|e_1\rangle$ such that $\langle e_1|e_1\rangle = \langle e_2|e_2\rangle = 2$
    $\langle e_3|e_3\rangle = 4$
    $\langle e_1|e_2\rangle = \imath\sqrt{2}$
    $\langle e_1|e_3\rangle = 1 + \imath$
    $\langle e_2|e_3\rangle = 2$

    Use Gram-Schmidt procedure to construct and orthonormal basis.

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19. If $\mathbf{A}$ is a Hermitian operator and $\mathbf{B}$ is anti-Hermitian then $\mathbf{AB} = 0$.

20. Prove that the unitary transformations leave the invariants of an operator unchanged, where the invariants are defined as

    $I_A = \text{tr A} = a_{ii}.$

    $II_A = \frac{1}{2}[(trA)^2 - trA^2]$

    $III_A = \frac{1}{6}[(trA)^3 + 2tr\ A^3 - 3(tr\ A^2)(tr\ A)] = \det[\text{A}]$

21. For Hermitian operators prove that

    $I_A = \Lambda_1 + \Lambda_2 + \Lambda_3$

    $II_A = \Lambda_1\Lambda_2 + \Lambda_2\Lambda_3 + \Lambda_3\Lambda_1$

    $III_A = \Lambda_1\Lambda_2\Lambda_3$

22. Find the eigenvalues and eigenvectors of the operator Ω whose elements in some basis are matrices:

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 4 \\ 0 & 4 & -3 \end{bmatrix}$$

23. Show that the product of unitary operators is unitary.

24. Show that the determinant of a unitary matrix is a complex number of unit modulus.

25. Verify that $R(\frac{\pi}{2}\hat{i})$ is unitary (orthogonal).

26. Consider the matrix

$$\Omega = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

    (a) Is it Hermitian?

    (b) Find its eigenvalues and eigenvectors.

    (c) Verify that $U^\dagger \Omega U$ is diagonal. U being the matrix of eigenvectors of Ω.

27. Consider the Hermitian matrix

$$\Omega = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

    (a) Show that $\omega_1 = \omega_2 = 1$; $\omega_3 = 2$.

    (b) Show that $|\omega = 2\rangle$ is any vector of the form

$$\frac{1}{(2a^2)^{1/2}}\begin{bmatrix} 0 \\ a \\ -a \end{bmatrix}$$

    (c) Show that the $\omega = 1$ eigenspace contains all vectors of the form

$$\frac{1}{(b^2 + 2c^2)^{1/2}}\begin{bmatrix} b \\ c \\ c \end{bmatrix}$$

28. An arbitrary $n \times n$ matrix need not have n eigenvectors. Consider as an example

$$\Omega = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}$$

   (a) Show that $\omega_1 = \omega_2 = 3$.

   (b) By feeding in this value show that only one eigenvector of the form

$$\frac{1}{(2a^2)^{1/2}} \begin{bmatrix} +a \\ -a \end{bmatrix}$$

We cannot find another one that is linearly independent.

29. Consider the matrix

$$\Omega = \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix}$$

   (a) Show that it is unitary.

   (b) Show that its eigenvalues are $e^{\imath\theta}$ and $e^{-\imath\theta}$.

   (c) Find the corresponding eigenvectors and show that they are orthogonal.

   (d) Verify that $U^{\dagger}\Omega U$ = diagonal matrix, where U is the matrix of eigenvectors of Ω.

30. Consider Hermitian matrices M^1, M^2, M^3, M^4 that obey

$$M^i M^j + M^j M^i = 2\delta^{ij} I$$

   (a) Show that the eigenvalues of M^i are ± 1.

   (b) By considering the relation

$$M^i M^j = -M^j M^i \text{ for } \mathrm{i} \neq \mathrm{j}$$

show that M^I are traceless.

   (c) Show that they cannot be odd-dimensional matrices.